SOME REMARKS ON LOCAL CLASS FIELD THEORY OF SERRE AND HAZEWINKEL

TAKASHI SUZUKI

Abstract. We give a new approach for local class field theory of Serre and Hazewinkel. In the case of characteristic zero, we also show a D-module version of this theory. Two-dimensional local class field theory is discussed in this framework.

1. Introduction

First we use the terminology of [Ser60], [Ser61], and [DG70] to state the first main theorem (Theorem 1.1) of this paper. Let k be a perfect field of characteristic $p \geq 0$ and $K = k((T))$. We fix an algebraic closure $\overline{K}$ of K. All the algebraic extensions of K are taken inside $\overline{K}$, for example, the separable closure K_s, the perfect closure K_p, the maximal abelian extension K^{ab}, the maximal unramified extension K^{ur}. The group of units of K can be viewed as a proalgebraic group over k in the sense of [Ser60]; we denote this group by $\mathbf{U}_K$. For each perfect k-algebra R (perfect means that the p-th power map is an isomorphism) we have the group

$$\mathbf{U}_K(R) = \left\{ \sum_{i=0}^{\infty} a_i \mathbf{T}^i \;\middle|\; a_i \in R,\ a_0 \in R^\times \right\}$$

of R-rational points. We consider the K_p-rational point $-T + \mathbf{T}$ of $\mathbf{U}_K$ and the corresponding morphism $\varphi\colon \operatorname{Spec} K_p \to \mathbf{U}_K$. We denote by η the composite map

$$I(K^{\mathrm{ab}}/K) \hookrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K) \xrightarrow{\sim} \pi_1^{\text{ét}}(\operatorname{Spec} K_p)^{\mathrm{ab}} \xrightarrow{\varphi} \pi_1^{\text{ét}}(\mathbf{U}_K)^{\mathrm{ab}} \twoheadrightarrow \pi_1^{k\text{-gp}}(\mathbf{U}_K).$$

Here we denote by $I(K^{\mathrm{ab}}/K)$ the inertia group of the extension K^{ab}/K, by $\pi_1^{\text{ét}}(\cdot)^{\mathrm{ab}}$ the maximal abelian quotient of the étale fundamental group, and by $\pi_1^{k\text{-gp}}$ the first left derived functor of the functor taking the maximal proconstant quotient in the category of commutative proalgebraic groups over k. Then we state the first main theorem of this paper:

Theorem 1.1. *The above defined map $\eta\colon I(K^{\mathrm{ab}}/K) \to \pi_1^{k\text{-gp}}(\mathbf{U}_K)$ is an isomorphism. Moreover, if k is either a finite field, an algebraic closure of a finite field, or a field of characteristic 0, then the inverse of η coincides with the isomorphism $\theta\colon \pi_1^{k\text{-gp}}(\mathbf{U}_K) \xrightarrow{\sim} I(K^{\mathrm{ab}}/K)$ of Serre-Hazewinkel (*[Ser61], [DG70]*).*

Next we assume that $\mathrm{char}(k) = 0$ (hence $K_p = K$) and use the notion of $\mathcal{D}$-module (cf. [HTT08]) to state the second main theorem (Theorem 1.2) of this paper. Let $n \geq 0$ be an integer and $\mathbf{U}_K^{n+1}$ be the proalgebraic group of $(n+1)$-th principal units. We say that a $\mathcal{D}$-module M on the k-scheme $\mathbf{U}_K/\mathbf{U}_K^{n+1}$ with $\mathcal{O}$-rank 1 is compatible with group structure if $\mu^* M \cong \mathrm{pr}_1^* M \otimes \mathrm{pr}_2^* M$, where $\mu\colon \mathbf{U}_K/\mathbf{U}_K^{n+1} \times \mathbf{U}_K/\mathbf{U}_K^{n+1} \to \mathbf{U}_K/\mathbf{U}_K^{n+1}$ is the multiplication and pr_i is the i-th projection $(i = 1, 2)$. A $\mathcal{D}$-module N on the k-scheme $\operatorname{Spec} K$ with $\mathcal{O}$-rank 1 is said to have irregularity n if its connection form with respect to some (hence any) K-basis of N has a form $f dT/T$ for some $f \in K^\times$ with valuation $-n$. With these terminologies the second main theorem of this paper is stated as follows:

Theorem 1.2. *Assume that* $\mathrm{char}(k) = 0$. *The map $\varphi\colon \operatorname{Spec} K \to \mathbf{U}_K$ induces, by pulling back, an equivalence of categories between the category $\mathcal{C}$ of $\mathcal{D}$-modules of $\mathcal{O}$-rank 1 on the k-scheme $\mathbf{U}_K/\mathbf{U}_K^{n+1}$ which are*

Date: March 18, 2009.
2000 *Mathematics Subject Classification.* Primary: 11S31; Secondary: 14F10.

compatible with group structure and the category $\mathcal{C}'$ of $\mathcal{D}$-modules of $\mathcal{O}$-rank 1 *on the k-scheme* $\operatorname{Spec} K$ *with irregularity* $\leq n$.

We also discuss a two-dimensional analogue of the above theory.

We give a couple of comments on literatures. First, the above defined map $\varphi\colon \operatorname{Spec} K_p \to \mathbf{U}_K$ have also been defined by Contou-Carrère ([CC94]). Second, the existence of an equivalence of categories between $\mathcal{C}$ and $\mathcal{C}'$ may be known for the specialists of the geometric Langlands correspondence (cf. [Fre07], [Bei06]).

Acknowledgements. This is the master thesis of the author at Kyoto University. The author would like to express his deep gratitude to his advisor Kazuya Kato and Tetsushi Ito for their suggestion of the problems, encouragement, and many helpful discussions.

2. Proof of Theorem 1.1

2.1. **Proof of the part "η is an isomorphism".** Both groups $\pi_1^{k\text{-gp}}(\mathbf{U}_K)$ and $I(K^{\mathrm{ab}}/K)$ are profinite abelian groups. Thus it is enough to show that η induces an isomorphism between the Pontryagin dual groups. The Pontryagin dual of $\pi_1^{k\text{-gp}}(\mathbf{U}_K)$ is canonically isomorphic to $\operatorname{Ext}_k^1(\mathbf{U}_K, \mathbb{Q}/\mathbb{Z})$ which is defined as the direct limit of the groups $\operatorname{Ext}_k^1(\mathbf{U}_K, \mathbb{Z}/m\mathbb{Z})$, $m \geq 1$, of extension classes of proalgebraic groups over k (cf. [Ser60, §5.4]). Therefore the problem is equivalent to showing that the dual map $\eta^\vee\colon \operatorname{Ext}_k^1(\mathbf{U}_K, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H^1(I(K^{\mathrm{ab}}/K), \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is an isomorphism for each prime number ℓ. Since $\mathbf{U}_K \cong \mathbf{G}_m \times \mathbf{U}_K^1$, we have

$$\operatorname{Ext}_k^1(\mathbf{U}_K, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong \operatorname{Ext}_k^1(\mathbf{G}_m, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \oplus \operatorname{Ext}_k^1(\mathbf{U}_K^1, \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

2.1.1. *The case $\ell \neq p$.* We compute the groups $\operatorname{Ext}_k^1(\mathbf{U}_K, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$, $H^1(I(K^{\mathrm{ab}}/K), \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ for $\ell \neq p$.

Lemma 2.1. *If $p = 0$, the usual exponential map $\prod_{n\geq 1} \mathbf{G}_a \to \mathbf{U}_K^1$ sending $(a_n)_{n\geq 1} \in \prod_{n\geq 1} \mathbf{G}_a$ to $\prod_{n\geq 1} \exp(a_n \mathbf{T}^n) \in \mathbf{U}_K^1$ is an isomorphism of proalgebraic groups. If $p > 0$, the Artin-Hasse exponential map $\prod_{p\nmid n\geq 1} W \to \mathbf{U}_K^1$ sending $a = (a_n)_{p\nmid n\geq 1} \in \prod_{p\nmid n\geq 1} W$ with $a_n = (a_{n0}, a_{n1}, \dots) \in W$ to $\prod_{p\nmid n\geq 1,\, m\geq 0} F(a_{nm}\mathbf{T}^{np^m}) \in \mathbf{U}_K^1$ is an isomorphism of proalgebraic groups. Here we denote by W the additive group of Witt vectors and set $F(t) = \exp(-\sum_{e\geq 0} t^{p^e}/p^e) \in \mathbb{Z}_p[[t]]$.*

Proof. See [Ser88, Chapter V, §3, 15 and 16]. □

Lemma 2.2. *The group $\operatorname{Ext}_k^1(\mathbf{U}_K^1, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is zero. The group $\operatorname{Ext}_k^1(\mathbf{G}_m, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is generated by the extension classes given by*

$$0 \longrightarrow \mathbb{Z}/\ell^d\mathbb{Z} \longrightarrow \mathbf{G}_m \xrightarrow{\ell^d} \mathbf{G}_m \longrightarrow 0,$$

where d runs through the integers such that $k^\times$ contains all the ℓ^d-th roots of unity and the map $\mathbb{Z}/\ell^d\mathbb{Z} \to \mathbf{G}_m$ corresponds to the choice of a primitive ℓ^d-th root of unity.

Proof. First we show that $\operatorname{Ext}_k^1(\mathbf{U}_K^1, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$. Lemma 2.1 shows that the ℓ-th power map induces an automorphism on $\mathbf{U}_K^1$. Since $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ is ℓ-power torsion, we have $\operatorname{Ext}_k^1(\mathbf{U}_K^1, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$.

Next we compute $\operatorname{Ext}_k^1(\mathbf{G}_m, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$. Since this group is isomorphic to the group of characters of ℓ-power order of $\pi_1^{k\text{-gp}}(\mathbf{U}_K)$, it is a union of subgroups $\operatorname{Ext}_k^1(\mathbf{G}_m, \mathbb{Z}/\ell^{d'}\mathbb{Z})$ for $d' \geq 1$. Taking the long exact sequence of the exact sequence

$$0 \longrightarrow \mu_{\ell^{d'}} \longrightarrow \mathbf{G}_m \xrightarrow{\ell^{d'}} \mathbf{G}_m \longrightarrow 0$$

we have an exact sequence

$$\operatorname{Hom}_k(\mathbf{G}_m, \mathbb{Z}/\ell^{d'}\mathbb{Z}) \longrightarrow \operatorname{Hom}_k(\mu_{\ell^{d'}}, \mathbb{Z}/\ell^{d'}) \longrightarrow \operatorname{Ext}_k^1(\mathbf{G}_m, \mathbb{Z}/\ell^{d'}\mathbb{Z}) \xrightarrow{\ell^{d'}} \operatorname{Ext}_k^1(\mathbf{G}_m, \mathbb{Z}/\ell^{d'}\mathbb{Z}). \tag{1}$$

Since $\mathbf{G}_m$ is connected and $\mathbb{Z}/\ell^{d'}\mathbb{Z}$ is discrete, the first term of (1) is zero. Since $\mathbb{Z}/\ell^{d'}\mathbb{Z}$ is killed by $\ell^{d'}$, the third map of (1) is a zero map. Thus we have an isomorphism $\operatorname{Hom}_k(\mu_{\ell^{d'}}, \mathbb{Z}/\ell^{d'}\mathbb{Z}) \xrightarrow{\sim} \operatorname{Ext}_k^1(\mathbf{G}_m, \mathbb{Z}/\ell^{d'}\mathbb{Z})$. If d is the maximal integer less than d' such that $k^\times$ contains all the ℓ^d-th roots of unity, then any morphism $\mu_{\ell^{d'}} \to \mathbb{Z}/\ell^{d'}\mathbb{Z}$ factors through the maximal constant quotient μ_{ℓ^d} of $\mu_{\ell^{d'}}$. Thus we have $\operatorname{Hom}_k(\mu_{\ell^{d'}}, \mathbb{Z}/\ell^{d'}\mathbb{Z}) = \operatorname{Hom}_k(\mu_{\ell^d}, \mathbb{Z}/\ell^d\mathbb{Z})$. If $d = d'$, the group $\operatorname{Hom}_k(\mu_{\ell^d}, \mathbb{Z}/\ell^d\mathbb{Z})$ is a cyclic group generated by an isomorphism $\mu_{\ell^d} \xrightarrow{\sim} \mathbb{Z}/\ell^d\mathbb{Z}$ corresponding to the choice of a primitive ℓ^d-th root of unity. This

generator corresponds to the desired extension class via the above defined isomorphism $\mathrm{Hom}_k(\mu_{\ell^d}, \mathbb{Z}/\ell^d\mathbb{Z}) \xrightarrow{\sim} \mathrm{Ext}^1_k(\mathbf{G}_m, \mathbb{Z}/\ell^d\mathbb{Z})$ because there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mu_{\ell^d} \arrow[r] \arrow[d, "\cong"] & \mathbf{G}_m \arrow[r, "\ell^d"] \arrow[d, "\mathrm{id}"] & \mathbf{G}_m \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \mathbb{Z}/\ell^d\mathbb{Z} \arrow[r] & \mathbf{G}_m \arrow[r, "\ell^d"] & \mathbf{G}_m \arrow[r] & 0,
\end{tikzcd}$$

where the map $\mathbb{Z}/\ell^d\mathbb{Z} \to \mathbf{G}_m$ is the inverse of the isomorphism $\mu_{\ell^d} \xrightarrow{\sim} \mathbb{Z}/\ell^d\mathbb{Z}$ followed by the inclusion $\mu_{\ell^d} \hookrightarrow \mathbf{G}_m$.

□

Lemma 2.3. *The group $H^1(I(K^{\mathrm{ab}}/K), \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is generated by the characters given by*

$$\sigma \mapsto \psi\big(\sigma((-T)^{1/\ell^d})/(-T)^{1/\ell^d}\big),$$

where d runs through the integers such that $k^\times$ contains all the ℓ^d-th roots of unity and $\psi\colon \mu_{\ell^d} \xrightarrow{\sim} \mathbb{Z}/\ell^d\mathbb{Z}$ is an isomorphism.

Proof. Note that $I(K^{\mathrm{ab}}/K) = \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{ab}} \cap K^{\mathrm{ur}}) \cong \mathrm{Gal}(K^{\mathrm{ab}}K^{\mathrm{ur}}/K^{\mathrm{ur}})$. For each integer $d \geq 1$, the field K^{ur} has a unique Galois extension of degree ℓ^d, namely $K^{\mathrm{ur}}((-T)^{1/\ell^d})$. This field is contained in $K^{\mathrm{ab}}K^{\mathrm{ur}}$ if and only if $k^\times$ contains all the ℓ^d-th roots of unity. □

Now we show that $\eta^\vee\colon \mathrm{Ext}^1_k(\mathbf{U}_K, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H^1(I(K^{\mathrm{ab}}/K), \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is an isomorphism for $\ell \neq p$. The extension class given in Lemma 2.2 gives an isogeny $\mathbf{G}_m \twoheadrightarrow \mathbf{G}_m$ with kernel $\mathbb{Z}/\ell^d\mathbb{Z}$. The map $\varphi\colon \mathrm{Spec}\, K_p \to \mathbf{U}_K$ followed by the projection $\mathbf{U}_K \twoheadrightarrow \mathbf{G}_m$ corresponds to the rational point $-T$. Taking the fiber product of these maps we have

$$\begin{tikzcd}
\mathrm{Spec}\, K_p((-T)^{1/\ell^d}) \arrow[r] \arrow[d] & \mathbf{G}_m \arrow[d, "\ell^d"] \\
\mathrm{Spec}\, K_p \arrow[r] & \mathbf{G}_m.
\end{tikzcd}$$

Thus, in view of the above lemmas, we know that the map $\eta^\vee\colon \mathrm{Ext}^1_k(\mathbf{U}_K, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H^1(I(K^{\mathrm{ab}}/K), \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is an isomorphism for $\ell \neq p$.

2.1.2. *The case $\ell = p$.* We have to treat the groups of characters of p-power order. We reduce the problem to that of order p.

Lemma 2.4. *Let $f\colon A \to B$ be a homomorphism between abelian groups A and B. If both A and B are p-divisible and p-power torsion, and f induces an isomorphism between the p-torsion part of A and that of B, then f is an isomorphism.*

Lemma 2.5. *The group $\mathrm{Ext}^1_k(\mathbf{G}_m, \mathbb{Q}_p/\mathbb{Z}_p)$ is zero. The group $\mathrm{Ext}^1_k(\mathbf{U}^1_K, \mathbb{Q}_p/\mathbb{Z}_p)$ is p-divisible.*

Proof. First we show that $\mathrm{Ext}^1_k(\mathbf{G}_m, \mathbb{Q}_p/\mathbb{Z}_p) = 0$. The p-th power map induces an automorphism on $\mathbf{G}_m$ since we work in the category of quasi-algebraic groups in the sense of [Ser60]. Since $\mathbb{Q}_p/\mathbb{Z}_p$ is p-power torsion, we have $\mathrm{Ext}^1_k(\mathbf{G}_m, \mathbb{Q}_p/\mathbb{Z}_p) = 0$.

Next we show the p-divisibility of $\mathrm{Ext}^1_k(\mathbf{U}^1_K, \mathbb{Q}_p/\mathbb{Z}_p)$. Since this group is isomorphic to the group of characters of p-power order of $\pi_1^{k\text{-gp}}(\mathbf{U}^1_K)$, it is a union of subgroups $\mathrm{Ext}^1_k(\mathbf{U}^1_K, \mathbb{Z}/p^d\mathbb{Z})$ for $d \geq 1$. We show that $\mathrm{Ext}^1_k(\mathbf{U}^1_K, \mathbb{Z}/p^d\mathbb{Z})$ is canonically isomorphic to $\bigoplus_{p \nmid n \geq 1} W_d(k)$ and the natural injection $\mathrm{Ext}^1_k(\mathbf{U}^1_K, \mathbb{Z}/p^d\mathbb{Z}) \hookrightarrow \mathrm{Ext}^1_k(\mathbf{U}^1_K, \mathbb{Z}/p^{d+1}\mathbb{Z})$ corresponds to the map $\bigoplus_{p \nmid n \geq 1} W_d(k) \hookrightarrow \bigoplus_{p \nmid n \geq 1} W_{d+1}(k)$ of multiplication by p, which imply the p-divisibility of $\mathrm{Ext}^1_k(\mathbf{U}^1_K, \mathbb{Q}_p/\mathbb{Z}_p)$. Since $\mathbf{U}^1_K \cong \prod_{p \nmid n \geq 1} W$, we have $\mathrm{Ext}^1_k(\mathbf{U}^1_K, \mathbb{Z}/p^d\mathbb{Z}) \cong \bigoplus_{p \nmid n \geq 1} \mathrm{Ext}^1_k(W, \mathbb{Z}/p^d\mathbb{Z})$. Taking the long exact sequence of the exact sequence

$$0 \longrightarrow W \xrightarrow{p^d} W \longrightarrow W_d \longrightarrow 0$$

we have an exact sequence

$$\mathrm{Hom}_k(W, \mathbb{Z}/p^d\mathbb{Z}) \longrightarrow \mathrm{Ext}^1_k(W_d, \mathbb{Z}/p^d) \longrightarrow \mathrm{Ext}^1_k(W, \mathbb{Z}/p^d\mathbb{Z}) \xrightarrow{p^d} \mathrm{Ext}^1_k(W, \mathbb{Z}/p^d\mathbb{Z}). \tag{2}$$

Since W is connected and $\mathbb{Z}/p^d\mathbb{Z}$ is discrete, the first term of (2) is zero. Since $\mathbb{Z}/p^d\mathbb{Z}$ is killed by p^d, the third map of (2) is a zero map. Thus we have an isomorphism $\mathrm{Ext}^1_k(W_d, \mathbb{Z}/p^d\mathbb{Z}) \xrightarrow{\sim} \mathrm{Ext}^1_k(W, \mathbb{Z}/p^d\mathbb{Z})$. There is a canonical element $\varepsilon_d \in \mathrm{Ext}^1_k(W_d, \mathbb{Z}/p^d\mathbb{Z})$ corresponding to the Artin-Schreier-Witt isogeny $\wp$. Each element $a \in W_d(k)$ gives, by multiplication, an endomorphism on W_d, hence an endomorphism a^* on $\mathrm{Ext}^1_k(W_d, \mathbb{Z}/p^d\mathbb{Z})$. The map $W_d(k) \to \mathrm{Ext}^1_k(W_d, \mathbb{Z}/p^d\mathbb{Z})$, $a \mapsto a^*\varepsilon_d$, is an isomorphism ([DG70, Chapter V, §3, 6.10]). Thus we get isomorphisms

$$\mathrm{Ext}^1_k(\mathbf{U}^1_K, \mathbb{Z}/p^d\mathbb{Z}) \cong \bigoplus_{p\nmid n\geq 1} \mathrm{Ext}^1_k(W, \mathbb{Z}/p^d\mathbb{Z}) \cong \bigoplus_{p\nmid n\geq 1} \mathrm{Ext}^1_k(W_d, \mathbb{Z}/p^d\mathbb{Z}) \cong \bigoplus_{p\nmid n\geq 1} W_d(k). \tag{3}$$

The natural injection $\mathrm{Ext}^1_k(\mathbf{U}^1_K, \mathbb{Z}/p^d\mathbb{Z}) \hookrightarrow \mathrm{Ext}^1_k(\mathbf{U}^1_K, \mathbb{Z}/p^{d+1}\mathbb{Z})$ corresponds, on each direct summand of the third term of (3), to the map $R^*p_*\colon \mathrm{Ext}^1_k(W_d, \mathbb{Z}/p^d\mathbb{Z}) \to \mathrm{Ext}^1_k(W_{d+1}, \mathbb{Z}/p^{d+1}\mathbb{Z})$, where $R\colon W_{d+1} \twoheadrightarrow W_d$ is the projection. The following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/p^d\mathbb{Z} \arrow[r] \arrow[d, "p"] & W_d \arrow[r, "\wp"] \arrow[d, "p"] & W_d \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & \mathbb{Z}/p^{d+1}\mathbb{Z} \arrow[r] & W_{d+1} \arrow[r, "\wp"] & W_{d+1} \arrow[r] & 0
\end{tikzcd}$$

shows that $p_*\varepsilon_d = p^*\varepsilon_{d+1}$. Hence $R^*p_*a^*\varepsilon_d = R^*a^*p^*\varepsilon_{d+1} = (pa)^*\varepsilon_{d+1}$. Thus the map $R^*p_*\colon \mathrm{Ext}^1_k(W_d, \mathbb{Z}/p^d\mathbb{Z}) \to \mathrm{Ext}^1_k(W_{d+1}, \mathbb{Z}/p^{d+1}\mathbb{Z})$ corresponds to the multiplication $p\colon W_d(k) \hookrightarrow W_{d+1}(k)$ via the third isomorphism of (3), as desired. □

Lemma 2.6. *The group $H^1(I(K^{\mathrm{ab}}/K), \mathbb{Q}_p/\mathbb{Z}_p)$ is p-divisible.*

Proof. The largest pro-p quotient of $\mathrm{Gal}(K_s/K)$ is pro-p free ([Ser02, Chapter I, §2.2, Corollary 1]). Thus $H^1(\mathrm{Gal}(K_s/K), \mathbb{Q}_p/\mathbb{Z}_p)$ is p-divisible. Since $H^1(I(K^{\mathrm{ab}}/K), \mathbb{Q}_p/\mathbb{Z}_p)$ is a quotient of $H^1(\mathrm{Gal}(K_s/K), \mathbb{Q}_p/\mathbb{Z}_p)$, the group $H^1(I(K^{\mathrm{ab}}/K), \mathbb{Q}_p/\mathbb{Z}_p)$ is also p-divisible. □

We calculate the groups of characters of order p.

Lemma 2.7. *As a special case ($d = 1$) of the isomorphism* (3)*, we have isomorphisms*

$$\mathrm{Ext}^1_k(\mathbf{U}^1_K, \mathbb{Z}/p\mathbb{Z}) \cong \bigoplus_{p\nmid n\geq 1} \mathrm{Ext}^1_k(W, \mathbb{Z}/p\mathbb{Z}) \cong \bigoplus_{p\nmid n\geq 1} \mathrm{Ext}^1_k(\mathbf{G}_a, \mathbb{Z}/p\mathbb{Z}) \cong \bigoplus_{p\nmid n\geq 1} k.$$

The map $k \to \mathrm{Ext}^1_k(\mathbf{G}_a, \mathbb{Z}/p\mathbb{Z})$ sends an element $a \in k^\times$ to the extension class given by

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbf{G}_a \xrightarrow{a^{-1}\wp} \mathbf{G}_a \longrightarrow 0,$$

where $\wp$ is the Artin-Schreier isogeny.

Proof. This is immediate from the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/p\mathbb{Z} \arrow[r] & \mathbf{G}_a \arrow[r, "\wp"] & \mathbf{G}_a \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z}/p\mathbb{Z} \arrow[r] \arrow[u, "\mathrm{id}"] & \mathbf{G}_a \arrow[r, "a^{-1}\wp"] \arrow[u, "\mathrm{id}"] & \mathbf{G}_a \arrow[r] \arrow[u, "a"] & 0.
\end{tikzcd}$$

□

Lemma 2.8. *The map defined by*

$$\bigoplus_{p\nmid n\geq 1} kT^{-n} \to H^1(I(K^{\mathrm{ab}}/K), \mathbb{Z}/p\mathbb{Z}),$$
$$aT^{-n} \mapsto \big(\sigma \mapsto \sigma(\wp^{-1}(aT^{-n})) - \wp^{-1}(aT^{-n})\big)$$

is an isomorphism.

Proof. Since the natural surjection $\mathrm{Gal}(K_s/K) \twoheadrightarrow \mathrm{Gal}(k_s/k)$ admits a section ([Ser02, Chapter II, §4.3, Exercises]), we know that the sequence

$$0 \longrightarrow H^1(\mathrm{Gal}(k_s/k), \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^1(\mathrm{Gal}(K_s/K), \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^1(I(K^{\mathrm{ab}}/K), \mathbb{Z}/p\mathbb{Z}) \longrightarrow 0$$

is exact. The first and the second term of this sequence is calculated by Artin-Schreier theory. Thus the third term also is calculated. The result is the desired form. □

Thus we are reduced to show that the map

$$\eta^\vee \colon \mathrm{Ext}^1_k(\mathbf{U}_K, \mathbb{Z}/p\mathbb{Z}) \cong \bigoplus_{p\nmid n\geq 1} k \to \bigoplus_{p\nmid n\geq 1} kT^{-n} \cong H^1(I(K^{\mathrm{ab}}/K), \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism. We need to calculate the following map:

$$\mathrm{Spec}\, K_p \xrightarrow{\varphi} \mathbf{U}^1_K/(\mathbf{U}^1_K)^p \cong \prod_{p\nmid n\geq 1} W/pW \cong \prod_{p\nmid n\geq 1} \mathbf{G}_a.$$

The map $\mathrm{Spec}\, K_p \to \mathbf{U}^1_K/(\mathbf{U}^1_K)^p$ corresponds to the K_p-rational point $1 - T^{-1}\mathbf{T}$ of $\mathbf{U}^1_K/(\mathbf{U}^1_K)^p$. The isomorphism $\prod_{p\nmid n\geq 1} \mathbf{G}_a \xrightarrow{\sim} \mathbf{U}^1_K/(\mathbf{U}^1_K)^p$ sends each element $(a_n)_{p\nmid n\geq 1}$ of the left hand side to $\prod_{p\nmid n\geq 1} F(a_n\mathbf{T}^n)$ of the right hand side.

Proposition 2.9. (1) *The inverse of the isomorphism $\prod_{p\nmid n\geq 1} \mathbf{G}_a \xrightarrow{\sim} \mathbf{U}^1_K/(\mathbf{U}^1_K)^p$ is given by the map*

$$\mathbf{U}^1_K/(\mathbf{U}^1_K)^p \xrightarrow{\mathrm{dlog}} \prod_{n\geq 1} \mathbf{G}_a\mathbf{T}^n \,\mathrm{dlog}\,\mathbf{T} \xrightarrow{\alpha} \prod_{p\nmid n\geq 1} \mathbf{G}_a,$$

where $\mathrm{dlog}(g) = (g'/g)d\mathbf{T}$ and $\alpha(\sum_{n\geq 1} b_n\mathbf{T}^n \,\mathrm{dlog}\,\mathbf{T}) = (-b_n/n)_{p\nmid n\geq 1}$.

(2) *The rational point $1 - T^{-1}\mathbf{T}$ corresponds to $(1/(nT^n))_{p\nmid n\geq 1}$ via the isomorphism $\mathbf{U}^1_K/(\mathbf{U}^1_K)^p(K_p) \cong \prod_{p\nmid n\geq 1} \mathbf{G}_a(K_p)$.*

(3) *The map $\mathrm{Spec}\, K_p \to \prod_{p\nmid n\geq 1} \mathbf{G}_a$ gives the K_p-rational point $(1/(nT^n))_{p\nmid n\geq 1}$ of $\prod_{p\nmid n\geq 1} \mathbf{G}_a$.*

Proof. (1): Using the identity $\mathrm{dlog}\, F(t) = -\sum_{e\geq 0} t^{p^e} \,\mathrm{dlog}\, t$, we have

$$\mathrm{dlog}\left(\prod_{p\nmid n\geq 1} F(a_n\mathbf{T}^n)\right) = -\sum_{\substack{e\geq 0\\ p\nmid n\geq 1}} (a_n\mathbf{T}^n)^{p^e} \,\mathrm{dlog}(a_n\mathbf{T}^n) = \sum_{\substack{e\geq 0\\ p\nmid n\geq 1}} (-n)(a_n\mathbf{T}^n)^{p^e} \,\mathrm{dlog}\,\mathbf{T}.$$

Thus the map $\alpha \circ \mathrm{dlog}$ sends $\prod_{p\nmid n\geq 1} F(a_n\mathbf{T}^n)$ to $(a_n)_{p\nmid n\geq 1}$, as desired. (2): A simple calculation shows that $(\alpha \circ \mathrm{dlog})(1 - T^{-1}\mathbf{T}) = (1/(nT^n))_{p\nmid n\geq 1}$. (3): This follows from (2). □

Now we calculate $\eta^\vee$. Let $n \geq 1$ be an integer prime to p and $a \neq 0$ be an element of k regarded as an element of $\bigoplus_{p\nmid n\geq 1} k$ by the n-th inclusion $k \hookrightarrow \bigoplus_{p\nmid n\geq 1} k$. The corresponding extension of $\mathbf{G}_a$ is given in Lemma 2.7. We have a cartesian diagram

$$\begin{tikzcd}
\mathrm{Spec}\, K_p(\wp^{-1}(a/nT^n)) \arrow[r] \arrow[d] & \mathbf{G}_a \arrow[d, "a^{-1}\wp"] \\
\mathrm{Spec}\, K_p \arrow[r] & \mathbf{G}_a.
\end{tikzcd}$$

Thus $\eta^\vee \colon \bigoplus_{p\nmid n\geq 1} k \to \bigoplus_{p\nmid n\geq 1} kT^{-n}$ preserves the direct factors and the map on the n-th factor is given by multiplication by $1/n$. This shows that $\eta \colon I(K^{ab}/K) \to \pi_1^{k\text{-gp}}(\mathbf{U}_K)$ is an isomorphism.

2.2. **Proof of the part "$\eta^{-1} = \theta$ for some cases".** First we show that $\eta^{-1} = \theta$ for the case where k is a finite field of q elements.

Proposition 2.10. *There is a cartesian diagram*

$$\begin{array}{ccc} \operatorname{Spec}(K_T^{\mathrm{ram}})_p & \longrightarrow & \mathbf{U}_K \\ \downarrow & & \downarrow{\scriptstyle F-1} \\ \operatorname{Spec} K_p & \xrightarrow[\varphi]{} & \mathbf{U}_K. \end{array}$$

Here K_T^{ram} is the field K adjoining all the T^m-torsion points (where m runs through the integers ≥ 1) of the Lubin-Tate formal group F_f (cf. [Iwa86]*) whose equation of formal multiplication by T is equal to $f(X) = TX + X^q$. The morphism $\operatorname{Spec}(K_T^{\mathrm{ram}})_p \to \mathbf{U}_K$ corresponds to the rational point $\sum_{m=0}^{\infty} \alpha_{m+1}\mathbf{T}^m$, where α_m is a generator of the module of T^m-torsion points of F_f. The map F is the q-th power relative Frobenius morphism (hence $F-1$ is the Lang isogeny over k). The induced isomorphism $\operatorname{Gal}(K_T^{\mathrm{ram}}/K) \cong U_K$ coincides with the one given by Lubin-Tate theory.*

Proof. We calculate the geometric fiber of $F-1$ over $-T+\mathbf{T}$. Let $g = \sum a_m \mathbf{T}^m$ be an element of $\mathbf{U}_K(\overline{K})$. The equation $F(g)/g = -T + \mathbf{T}$ is equivalent to the system of equations $f(a_0) = 0$, $f(a_{m+1}) = a_m$, $m \geq 0$. Thus, for each $m \geq 0$, a_m is a generator of the module of T^{m+1}-torsion points of F_f. This proves the existence of the above cartesian diagram. Next we calculate the action of $\operatorname{Gal}(K_T^{\mathrm{ram}}/K)$ on the fiber of $F-1$ over $-T+\mathbf{T}$. The Lubin-Tate group F_f for $f(X) = TX + X^q$ is the formal completion $\widehat{\mathbf{G}}_a$ of the additive group with the formal multiplication of each element $\sum b_m T^m$ of $\mathcal{O}_K$ being given by the power series $\sum b_m f^{\circ m}(X) \in \operatorname{End} \widehat{\mathbf{G}}_a$, where $f^{\circ m}$ is the m-th iteration of f. Thus, if σ corresponds to $u(T) = \sum b_m T^m$ via the isomorphism $\operatorname{Gal}(K_T^{\mathrm{ram}}/K) \cong U_K$ of Lubin-Tate theory, we have

$$\sigma\left(\sum_{m\geq 0} \alpha_{m+1}\mathbf{T}^m\right) = \sum_{m \geq 0} \sigma(\alpha_{m+1})\mathbf{T}^m = \sum_{0 \leq k \leq m < \infty} b_k \alpha_{m+1-k}\mathbf{T}^m = u(\mathbf{T}) \sum_{m\geq 0} \alpha_{m+1}\mathbf{T}^m.$$

Thus the action of σ on the fiber of $F-1$ over $-T+\mathbf{T}$ is given by multiplication by $u(\mathbf{T})$, as required. □

Thus the map $\eta\colon I(K^{\mathrm{ab}}/K) \to \pi_1^{k\text{-gp}}(\mathbf{U}_K)$ factors through the isomorphism of Lubin-Tate theory:

$$I(K^{\mathrm{ab}}/K) \to \operatorname{Gal}(K_T^{\mathrm{ram}}/K) \cong U_K \cong \pi_1^{k\text{-gp}}(\mathbf{U}_K).$$

Since the isomorphism θ of Serre-Hazewinkel for finite k coincides with the one given by Lubin-Tate theory, the equality $\eta^{-1} = \theta$ for such k follows.

Remark. The above proposition, combined with the fact that $\eta\colon I(K^{\mathrm{ab}}/K) \to \pi_1^{k\text{-gp}}(\mathbf{U}_K)$ is an isomorphism, which was proved in the previous subsection, gives another proof of the local Kronecker-Weber theorem for Lubin-Tate extensions: we have just been proved that the canonical surjection $I(K^{\mathrm{ab}}/K) \twoheadrightarrow \operatorname{Gal}(K_T^{\mathrm{ram}}/K)$ is an isomorphism, that is, $K^{\mathrm{ab}} = K_T^{\mathrm{ram}} K^{\mathrm{ur}}$.

Next we show that $\eta^{-1} = \theta$ for the case where k is an algebraic closure of a finite field. We put $K_n = \mathbb{F}_{p^n}((T))$. Then we have

$$\operatorname{Gal}(K^{\mathrm{ab}}/K) = \operatorname{Gal}((\cup K_n)^{\mathrm{ab}}/\cup K_n) = \varprojlim I(K_n^{\mathrm{ab}}/K_n).$$

Also by [DG70, Chapter V, §3, 2.3] we have $\pi_1^{k\text{-gp}}(\mathbf{U}_K) = \lim_{\leftarrow} \pi_1^{k\text{-gp}}(\mathbf{U}_{K_n})$. Since the maps $\varphi\colon \operatorname{Spec}(K_n)_p \to \mathbf{U}_{K_n}$ are compatible with base extension, the equality $\eta^{-1} = \theta$ is reduced to the finite residue field case.

Finally we treat the case $\operatorname{char}(k) = 0$. Let L/K be a totally ramified abelian extension of degree n. Kummer theory and the exponential map show that the inclusion $\mathbf{U}_K \hookrightarrow \mathbf{U}_L$ induces an isomorphism $\mathbf{U}_K \xrightarrow{\sim} \mathbf{U}_L/\mathbf{V}_{L/K}$, where $\mathbf{V}_{L/K}$ is a subgroup of $\mathbf{U}_K$ generated by $(\sigma-1)\mathbf{U}_L$ for various $\sigma \in \operatorname{Gal}(L/K)$. The composite of this isomorphism and the norm map $N_{L/K}\colon \mathbf{U}_L/\mathbf{V}_{L/K} \to \mathbf{U}_K$ is the n-th power endomorphism

on $\mathbf{U}_K$, which is an automorphism on the subgroup $\mathbf{U}_K^1$. Thus we have the following diagram whose two squares are both cartesian:

$$\begin{array}{ccccc} \operatorname{Spec} K^{\mathrm{ur}}((-T)^{1/n}) & \longrightarrow & \mathbf{U}_L/\mathbf{V}_{L/K} & \longrightarrow & \mathbf{G}_m \\ \big\downarrow & & \big\downarrow{\scriptstyle N_{L/K}} & & \big\downarrow{\scriptstyle n} \\ \operatorname{Spec} K^{\mathrm{ur}} & \longrightarrow & \mathbf{U}_K & \longrightarrow & \mathbf{G}_m. \end{array}$$

Then the equality $\eta^{-1} = \theta$ follows.

3. Proof of Theorem 1.2

First we describe the category $\mathcal{C}$. Write $\mathbf{U}_K/\mathbf{U}_K^{n+1} \cong \mathbf{G}_m \times \mathbf{G}_a^n = \operatorname{Spec} k[T_0^{\pm}, T_1, \dots, T_n]$ and put $A = k[T_0^{\pm}, T_1, \dots, T_n]$, $D_A = A[\partial_{T_0}, \dots, \partial_{T_n}]$. Since A is a UFD, any line bundle on $\mathbf{U}_K/\mathbf{U}_K^{n+1}$ can be trivialized. Let $M = Ae^M$ be a D_A-module of A-rank 1 with a basis e^M and a connection form $\omega^M = f_0^M dT_0/T_0 + \sum_{1\le i\le n} f_i^M dT_i$, where $f_i^M \in A$. For M to be compatible with group structure, it is necessary and sufficient that f_i^M is equal to a constant $a_i^N \in k$ for each i. Let $N = Ae^N$ be another D_A-module of A-rank 1 with a connection form $\omega^N = a_0^N dT_0/T_0 + \sum_{1\le i\le n} a_i^N dT_i$, $a_i^N \in k$. We determine the space of D_A-homomorphisms $\operatorname{Hom}_{D_A}(M, N)$. Since both M and N are A-rank 1, this space can be viewed as a k-subspace of A. An element $g \in \operatorname{Hom}_{D_A}(M, N) \subset A$ should satisfy a system of differential equations

$$\partial_{T_0} g = \frac{a_0^M - a_0^N}{T_0} g, \quad \partial_{T_i} g = (a_i^M - a_i^N) g, \quad 1 \le i \le n.$$

This system has a non-zero solution g in A if and only if $a_0^M - a_0^N \in \mathbb{Z}$ and $a_i^M = a_i^N$ for $1 \le i \le n$. If these conditions are satisfied, the space of solutions is a 1-dimensional k-vector space spanned by $T_0^{a_0^M - a_0^N}$. In particular the isomorphism classes of objects of the category $\mathcal{C}$ is classified by the space

$$(k/\mathbb{Z}) \frac{dT_0}{T_0} \oplus \bigoplus_{1\le i\le n} k dT_i$$

by taking the connection form.

Next we describe the category $\mathcal{C}'$. If $M = Ke^M$ (resp. $N = Ke^N$) is a $D_K = K[\partial_T]$-module with irregularity $\le n^M$ (resp. $\le n^N$) with a connection form $f^M dT/T = \sum_{-\infty<i\le n^M} a_i^M T^{-i} dT/T$ (resp. $f^N dT = \sum_{-\infty<i\le n^N} a_i^N T^{-i} dT/T$), then the space $\operatorname{Hom}_{D_K}(M, N)$ is zero unless $a_0^M - a_0^N \in \mathbb{Z}$ and $a_i^M = a_i^N$ for $i \ge 1$. If these conditions are satisfied, $\operatorname{Hom}_{D_K}(M, N)$ is a 1-dimensional k-vector space spanned by

$$T^{a_0^M - a_0^N} \exp\left(\sum_{i<0} \frac{a_i^M - a_i^N}{-i} T^{-i}\right).$$

In particular the isomorphism classes of objects of the category $\mathcal{C}'$ is classified by the space

$$\left((k/\mathbb{Z}) \oplus \bigoplus_{1\le i\le n} kT^{-i}\right) \frac{dT}{T}$$

by taking the connection form.

Now we describe the functor of pulling back by φ: $\operatorname{Spec} K \to \mathbf{U}_K/\mathbf{U}_K^{n+1}$. The map φ followed by the isomorphism $\mathbf{U}_K/\mathbf{U}_K^{n+1} \cong \mathbf{G}_m \times \mathbf{G}_a^n$ given in Lemma 2.1 corresponds to a rational point $(-T, (-T^{-i}/i)_i)$. If M is an object of $\mathcal{C}$ with a connection form $\omega^M = a_0^M dT_0/T_0 + \sum_{1\le i\le n} a_i^M dT_i$, then the pullback $\varphi^* M$ has a connection form $\varphi^* \omega^M = \sum_{0\le i\le n} a_i^M T^{-i} dT/T$. Using this description and the above classification we know that the functor of pulling back by φ is fully faithful and essentially surjective. Thus we get Theorem 1.2.

4. An auxiliary result

The following proposition is a refinement of Proposition 2.10.

Proposition 4.1. *Assume that k is either a finite field, an algebraic closure of a finite field, or a field of characteristic 0. Then, for any finite totally ramified abelian extension L/K, there is a map $\operatorname{Spec} L_p^{\mathrm{ur}} \to \mathbf{U}_L/\mathbf{V}_{L/K}$ and a cartesian diagram*

$$\begin{tikzcd}
\operatorname{Spec} L_p^{\mathrm{ur}} \arrow[r] \arrow[d] & \mathbf{U}_L/\mathbf{V}_{L/K} \arrow[d, "N_{L/K}"] \\
\operatorname{Spec} K_p^{\mathrm{ur}} \arrow[r, "\varphi"'] & \mathbf{U}_K.
\end{tikzcd}$$

The induced isomorphism $\operatorname{Gal}(L/K) \cong \operatorname{Ker}(N_{L/K})$ coincides with θ.

We prove this proposition below. Note that the group $\mathbf{U}_L(\overline{K})$ is equipped with two different actions of $\operatorname{Gal}(K_s/K)$, namely the one induced by the action of $\operatorname{Gal}(L/K)$ on the proalgebraic group $\mathbf{U}_L$ and the one induced by the action on the coefficient field $\overline{K}$. For $g \in \mathbf{U}_L(\overline{K})$ and $\sigma \in \operatorname{Gal}(K_s/K)$ we denote by $g^{[\sigma]}$ (resp. g^{σ}) the action of $\sigma \in \operatorname{Gal}(K_s/K)$ on $g \in \mathbf{U}_L(\overline{K})$ in the former (resp. the latter) sense.

Lemma 4.2. *Assume that k is finite. Let $m \geq 1$ be an integer and $L = K_T^m$ be the field K adjoining all the T^m-torsion points of the Lubin-Tate group F_f. For any $g \in \mathbf{U}_L$ the image of $N_{L/K} g$ in $\mathbf{U}_K/\mathbf{U}_K^m$ depends only on the image of $g^{F-1} = g^F/g$ in $\mathbf{U}_L/\mathbf{V}_{L/K}$. Thus we obtain a map $N_{L/K} \circ (F-1)^{-1} \colon \mathbf{U}_L/\mathbf{V}_{L/K} \to \mathbf{U}_K/\mathbf{U}_K^m$. This map makes the following diagram commutative*:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Gal}(L/K) \arrow[r] \arrow[d] & \mathbf{U}_L/\mathbf{V}_{L/K} \arrow[r, "N_{L/K}"] \arrow[d, "N_{L/K}\circ(F-1)^{-1}"] & \mathbf{U}_K \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & U_K/U_K^m \arrow[r] & \mathbf{U}_K/\mathbf{U}_K^m \arrow[r, "F-1"'] & \mathbf{U}_K/\mathbf{U}_K^m \arrow[r] & 0.
\end{tikzcd}$$

Here the map $\operatorname{Gal}(L/K) \to \mathbf{U}_L/\mathbf{V}_{L/K}$ is given by $\sigma \mapsto \boldsymbol{\alpha}_m^{[\sigma]-1} = \boldsymbol{\alpha}_m^{[\sigma]}/\boldsymbol{\alpha}_m$ ($\boldsymbol{\alpha}_m$ is defined similarly to $\mathbf{T}$) and the map $\operatorname{Gal}(L/K) \to U_K/U_K^m$ is the isomorphism of local class field theory. All other unnamed maps are the canonical ones.

Proof. The well-definedness of $N_{L/K} \circ (F-1)^{-1}$: The kernel of the endomorphism $F-1$ of $\mathbf{U}_L/\mathbf{V}_{L/K}$ is equal to $U_L \mathbf{V}_{L/K}/\mathbf{V}_{L/K}$; its image by $N_{L/K}$ is contained in $N_{L/K}(U_L) = U_K^m$. This proves the well-definedness.

The commutativity of the left square: See [Ser79, Chapter XIII, §5]. □

Proof of Proposition 4.1. First we prove Proposition 4.1 for the case where k is finite and $L = K_T^m$. By Proposition 2.10 we have a cartesian diagram

$$\begin{tikzcd}
\operatorname{Spec} L_p^{\mathrm{ur}} \arrow[r] \arrow[d] & \mathbf{U}_K/\mathbf{U}_K^m \arrow[d, "F-1"] \\
\operatorname{Spec} K_p^{\mathrm{ur}} \arrow[r, "\varphi"'] & \mathbf{U}_K/\mathbf{U}_K^m.
\end{tikzcd}$$

Combining this diagram with Lemma 4.2 we have the following diagram whose two squares are both cartesian:

$$\begin{tikzcd}
\operatorname{Spec} L_p^{\mathrm{ur}} \arrow[r] \arrow[d] & \mathbf{U}_L/\mathbf{V}_{L/K} \arrow[r, "N_{L/K}\circ(F-1)^{-1}"'] \arrow[d, "N_{L/K}"] & \mathbf{U}_K/\mathbf{U}_K^m \arrow[d, "F-1"] \\
\operatorname{Spec} K_p^{\mathrm{ur}} \arrow[r, "\varphi"'] & \mathbf{U}_K \arrow[r] & \mathbf{U}_K/\mathbf{U}_K^m.
\end{tikzcd}$$

This diagram induces isomorphisms $\operatorname{Gal}(L/K) \cong \operatorname{Ker}(N_{L/K}) \cong U_K/U_K^m$. Since this induced isomorphism $\operatorname{Gal}(L/K) \cong U_K/U_K^m$ (resp. $\operatorname{Ker}(N_{L/K}) \cong U_K/U_K^m$) coincides with the one given by local class field theory by Proposition 2.10 (resp. by Lemma 4.2), so is $\operatorname{Gal}(L/K) \cong \operatorname{Ker}(N_{L/K})$.

Now let L/K be an arbitrary finite totally ramified abelian extension. By the local Kronecker-Weber theorem there exists an integer m such that $L^{\mathrm{ur}} \subset (K_T^m)^{\mathrm{ur}}$. Consider the following diagram whose two squares are both cartesian:

$$\begin{array}{ccccc} \operatorname{Spec}(K_T^m)_p^{\mathrm{ur}} & \longrightarrow & X & \longrightarrow & \operatorname{Spec} K_p^{\mathrm{ur}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{U}_{K_T^m}/\mathbf{V}_{K_T^m/K} & \xrightarrow[N_{K_T^m/L}]{} & \mathbf{U}_L/\mathbf{V}_{L/K} & \xrightarrow[N_{L/K}]{} & \mathbf{U}_K. \end{array}$$

Since the maps $\operatorname{Spec}(K_T^m)_p^{\mathrm{ur}} \to X \to \operatorname{Spec} K_p^{\mathrm{ur}}$ are finite étale, the scheme X is of the form $\operatorname{Spec} L'$ for some intermediate extension L' of $(K_T^m)_p^{\mathrm{ur}}/K_p^{\mathrm{ur}}$. We show that $L' = L_p^{\mathrm{ur}}$. Let g be an element of the fiber of $N_{K_T^m/K}$ over $-T+\mathbf{T}$ and put $h = N_{K_T^m/L}g$ and $\alpha = N_{K_T^m/L}\alpha_m$. Then h is the rational point corresponding the map $\operatorname{Spec} L' \to \mathbf{U}_L/\mathbf{V}_{L/K}$ and α is a prime element of L. For any $\sigma \in \operatorname{Gal}(L/K)$ the equality $g^{\sigma-1} = \boldsymbol{\alpha}_m^{[\sigma]-1}$ holds in $\mathbf{U}_L/\mathbf{V}_{L/K}$ by Proposition 4.1 for the extension K_T^m/K. Taking $N_{K_T^m/L}$ on both side of this equality we have $h^{\sigma-1} = \boldsymbol{\alpha}^{[\sigma]-1}$. Since $\sigma|_{L'} = 1$ (resp. $\sigma|_L = 1$) is equivalent to $h^{\sigma-1} = 1$ (resp. $\boldsymbol{\alpha}^{[\sigma]-1} = 1$), we have $L' = L$. This proves Proposition 4.1 for finite k.

The proof of Proposition 4.1 for the case where k is an algebraic closure of a finite field is reduced to the finite case by the similar argument used in the proof of $\eta^{-1} = \theta$ for such k. The case $\operatorname{char}(k) = 0$ is already treated in the proof of $\eta^{-1} = \theta$ for the characteristic 0 case.

□

5. A two-dimensional analogue

In this section we discuss an analogue of the above theory for the field $K = k((S))((T))$. We denote by K_2 the functor of the second algebraic K-group ([Bas73]). For each perfect k-algebra R we have an abelian group $K_2(R[[\mathbf{S},\mathbf{T}]])$. This gives a group functor which we denote by $K_2[[\mathbf{S},\mathbf{T}]]$. The K_p-rational point

$$\{-S+\mathbf{S}, -T+\mathbf{T}\} \in K_2[[\mathbf{S},\mathbf{T}]](K_p) = K_2\big(k((S))((T))_p[[\mathbf{S},\mathbf{T}]]\big)$$

gives a morphism $\varphi\colon \operatorname{Spec} K_p \to K_2[[\mathbf{S},\mathbf{T}]]$, where $\{\cdot,\cdot\}$ denotes the symbol map. This is an analogue of the map $\operatorname{Spec} k((T))_p \to \mathbf{U}_{k((T))}$ previously defined and studied.

When k is a finite field $\mathbb{F}_q$, the field $K = k((S))((T))$ is called a two-dimensional local field ([FK00]) of positive characteristic. For each perfect k-algebra R there is a k-automorphism of $R[[\mathbf{S},\mathbf{T}]]$ which maps each element of R to its q-th power and fixes $\mathbf{S}$ and $\mathbf{T}$. This k-automorphism induces an k-automorphism on $K_2[[\mathbf{S},\mathbf{T}]]$ which we denote by F. Consider the following cartesian diagram:

$$\begin{array}{ccc} X & \longrightarrow & K_2[[\mathbf{S},\mathbf{T}]] \\ \downarrow & & \downarrow{\scriptstyle F-1} \\ \operatorname{Spec} K_p & \xrightarrow{\varphi} & K_2[[\mathbf{S},\mathbf{T}]]. \end{array}$$

Then we expect that $K_2[[\mathbf{S},\mathbf{T}]]$ can be viewed as a kind of "algebraic group over k" and the equation $x^{F-1} = \{-S+\mathbf{S}, -T+\mathbf{T}\}$ gives a two-dimensional analogue of Lubin-Tate theory so that X is the Spec of the perfect closure of a large totally ramified abelian extension of K (cf. Proposition 2.10).

To avoid some technical difficulties and prove a rigorous statement, we use the space of 2-forms instead of $K_2[[\mathbf{S},\mathbf{T}]]$. For each perfect k-algebra R we have the space of 2-forms $\Omega^2_{R[[\mathbf{S},\mathbf{T}]]/R}$. This functor is represented by a proalgebraic group over k isomorphic to an infinite product of $\mathbf{G}_a$ with coordinate $z_{ij} := \mathbf{S}^i\mathbf{T}^j d\mathbf{S} \wedge d\mathbf{T}$, $i, j \geq 0$. We denote this group by $\Omega_{[[\mathbf{S},\mathbf{T}]]}$. The dlog map $K_2[[\mathbf{S},\mathbf{T}]] \to \Omega_{[[\mathbf{S},\mathbf{T}]]}$ is defined. There is a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & K_2[[\mathbf{S},\mathbf{T}]] & \xrightarrow[\mathrm{dlog}]{} & \Omega_{[[\mathbf{S},\mathbf{T}]]} \\ \downarrow & & \downarrow{\scriptstyle F-1} & & \downarrow{\scriptstyle F-1} \\ \operatorname{Spec} K_p & \xrightarrow[\varphi]{} & K_2[[\mathbf{S},\mathbf{T}]] & \xrightarrow[\mathrm{dlog}]{} & \Omega_{[[\mathbf{S},\mathbf{T}]]}. \end{array}$$

We put $\varphi' = \operatorname{dlog} \circ \varphi$.

Proposition 5.1. *There is a cartesian diagram*

$$\begin{array}{ccc} \operatorname{Spec} A_p & \longrightarrow & \Omega_{[[\mathbf{S},\mathbf{T}]]} \\ \big\downarrow & & \big\downarrow{\scriptstyle F-1} \\ \operatorname{Spec} K_p & \xrightarrow[\varphi']{} & \Omega_{[[\mathbf{S},\mathbf{T}]]}. \end{array}$$

Here we denote by A the ring $K[x_{ij} \,|\, i, j \geq 0]/(x_{ij}^q - x_{ij} - S^{-i-1}T^{-j-1})$ and by A_p the direct limit of the p-th power maps $A \to A \to \cdots$.

Proof. The map $\varphi' \colon \operatorname{Spec} K_p \to \Omega_{[[\mathbf{S},\mathbf{T}]]}$ corresponds to a rational point

$$\operatorname{dlog}\{-S + \mathbf{S}, -T + \mathbf{T}\} = \frac{d(-S + \mathbf{S})}{-S + \mathbf{S}} \wedge \frac{d(-T + \mathbf{T})}{-T + \mathbf{T}} = \sum_{i,j \geq 0} S^{-i-1}T^{-j-1}\mathbf{S}^i\mathbf{T}^j d\mathbf{S} \wedge d\mathbf{T}.$$

If $\sum x_{ij}\mathbf{S}^i\mathbf{T}^j d\mathbf{S} \wedge d\mathbf{T} \in \Omega_{[[\mathbf{S},\mathbf{T}]]}(\overline{K})$ lies in the geometric fiber of $F - 1$ over this rational point, it should satisfy

$$(F - 1) \sum_{i,j \geq 0} x_{ij}\mathbf{S}^i\mathbf{T}^j d\mathbf{S} \wedge d\mathbf{T} = \sum_{i,j \geq 0} (x_{ij}^q - x_{ij})\mathbf{S}^i\mathbf{T}^j d\mathbf{S} \wedge d\mathbf{T} = \sum_{i,j \geq 0} S^{-i-1}T^{-j-1}\mathbf{S}^i\mathbf{T}^j d\mathbf{S} \wedge d\mathbf{T}.$$

Thus we get the proposition. □

Next we assume $\operatorname{char}(k) = 0$ and calculate the pullback of a $\mathcal{D}$-module on $\Omega_{[[\mathbf{S},\mathbf{T}]]}$ which is compatible with group structure in analogy with Theorem 1.2. Let $n, m \geq 0$ be integers. The group $\Omega_{[[\mathbf{S},\mathbf{T}]]}$ has the algebraic quotient G with coordinate z_{ij}, $1 \leq i \leq n$, $1 \leq j \leq m$. Any $\mathcal{D}$-module M on G of $\mathcal{O}$-rank 1 which is compatible with group structure has a connection form of the form $\sum a_{ij} dz_{ij}$ with $a_{ij} \in k$. We have

$$\begin{aligned} (\varphi')^* \sum_{\substack{0 \leq i \leq n \\ 0 \leq j \leq m}} a_{ij} dz_{ij} &= \sum_{\substack{0 \leq i \leq n \\ 0 \leq j \leq m}} a_{ij} d(S^{-i-1}T^{-j-1}) \\ &= - \sum_{\substack{0 \leq i \leq n \\ 0 \leq j \leq m}} (i + 1)a_{ij}S^{-i-2}T^{-j-1}dS - \sum_{\substack{0 \leq i \leq n \\ 0 \leq j \leq m}} (j + 1)a_{ij}S^{-i-1}T^{-j-2}dT. \end{aligned}$$

This is a connection form of the pullback of M by φ'.

References

[Bas73] Hyman Bass, editor. *Algebraic K-theory. I: Higher K-theories.* Lecture Notes in Mathematics, Vol. 341. Springer-Verlag, Berlin, 1973.

[Bei06] Alexander Beilinson. Langlands parameters for Heisenberg modules. In *Studies in Lie theory*, volume 243 of *Progr. Math.*, pages 51–60. Birkhäuser Boston, Boston, MA, 2006.

[CC94] Carlos Contou-Carrère. Jacobienne locale, groupe de bivecteurs de Witt universel, et symbole modéré. *C. R. Acad. Sci. Paris Sér. I Math.*, 318(8):743–746, 1994.

[DG70] Michel Demazure and Pierre Gabriel. *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs.* Masson & Cie, Éditeur, Paris, 1970. (Appendix: M. Hazewinkel, Corps de classes local).

[FK00] Ivan Fesenko and Masato Kurihara, editors. *Invitation to higher local fields*, volume 3 of *Geometry & Topology Monographs.* Geometry & Topology Publications, Coventry, 2000. Papers from the conference held in Münster, August 29–September 5, 1999.

[Fre07] Edward Frenkel. *Langlands correspondence for loop groups*, volume 103 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2007.

[HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki. *D-modules, perverse sheaves, and representation theory*, volume 236 of *Progress in Mathematics.* Birkhäuser Boston Inc., Boston, MA, 2008. Translated from the 1995 Japanese edition by Takeuchi.

[Iwa86] Kenkichi Iwasawa. *Local class field theory.* Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1986. Oxford Mathematical Monographs.

[Ser60] Jean-Pierre Serre. Groupes proalgébriques. *Inst. Hautes Études Sci. Publ. Math.*, 7, 1960.

[Ser61] Jean-Pierre Serre. Sur les corps locaux à corps résiduel algébriquement clos. *Bull. Soc. Math. France*, 89:105–154, 1961.

[Ser79] Jean-Pierre Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1979. Translated from the French by Marvin Jay Greenberg.

[Ser88] Jean-Pierre Serre. *Algebraic groups and class fields*, volume 117 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1988. Translated from the French.

[Ser02] Jean-Pierre Serre. *Galois cohomology*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, English edition, 2002. Translated from the French by Patrick Ion and revised by the author.

Department of Mathematics, Kyoto University, Kyoto, 606-8502, Japan
E-mail address: suzuki@math.kyoto-u.ac.jp